IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

A criterion for local interconvertibility between all-tripartite Gaussian states

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2004 J. Phys. A: Math. Gen. 37 10699

(http://iopscience.iop.org/0305-4470/37/44/017)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.64
The article was downloaded on 02/06/2010 at 19:31

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **37** (2004) 10699–10710 PII: S0305-4470(04)79969-2

A criterion for local interconvertibility between all-tripartite Gaussian states

Liang Wang[1,2], **Shu-Shen Li**[1] **and YiJing Yan**[2]

[1] State Key Laboratory for Superlattices and Microstructures, Institute of Semiconductors, Chinese Academy of Sciences, PO Box 912, Beijing 100083, People's Republic of China
[2] Department of Chemistry, Hong Kong University of Science and Technology, Kowloon, Hong Kong

E-mail: wangliang@red.semi.ac.cn

Received 25 April 2004, in final form 26 April 2004
Published 20 October 2004
Online at stacks.iop.org/JPhysA/37/10699
doi:10.1088/0305-4470/37/44/017

Abstract

We arrive at a necessary and sufficient criterion that can be readily used for interconvertibility between general, all-tripartite Gaussian states under local quantum operation. The derivation involves a systematic reduction that converts the original complex conditions in high-dimensional, $6n \times 6n$ matrix space eventually into 2×2 matrix problems.

PACS numbers: 03.67.−a, 03.65.Ud, 42.50.Lc

1. Introduction

Continuous variable systems, due to the immense achievements especially on the quantum optics manipulation of their entangled Gaussian states, have become an important component of quantum information [1]. All Gaussian states can be prepared starting from coherent states by experimental means such as beam splitters, phase shifters and squeezers. Teleportation schemes for continuous variables have been both theoretically proposed and experimentally implemented [2]. The problems of separability, quantification and distillability for Gaussian states have been basically solved [3]. Entanglement of formation for symmetric Gaussian states has been discussed recently [4]. Another advantage of exploiting continuous variable systems in quantum information processing is the mathematical simplicity of Gaussian states that support a complete characterization of entanglement transformation under local quantum operations. The first step towards the theory of entanglement transformation was taken by Eisert and Plenio [5]. They arrive at the general form of the necessary and sufficient criterion for the interconvertibility between Gaussian states, one mode in each location. Recently, Giedke *et al* [6] have constructed a theory of entanglement transformation for all-bipartite Gaussian pure states under actions of local quantum operations and classical communication.

0305-4470/04/4410699+12$30.00 © 2004 IOP Publishing Ltd Printed in the UK

Wang[3] *et al* [7, 8] have extended the Eisert–Plenio scheme to more complicated systems. In [7] a special form of covariance matrix for the simplest tripartite Gaussian system, with one mode in each location, was first constructed, and then used to arrive at the necessary and sufficient criterion for the interconvertibility between the special tripartite-entangled Gaussian states in the study. The resulting criterion shows that the interconvertible conditions include not only inequalities but also equalities. Reported subsequently in [8] was also the necessary and sufficient criterion for all-bipartite Gaussian states. In this paper, we extend our previous work [7, 8] to all-tripartite Gaussian states, also called an $n \times n \times n$ system, in which there are n modes at each of Alice, Bob and Charlie's locations, and the Gaussians include both pure and mixed states. Again, by virtue of the normal form of covariance matrix to be constructed in section 2 for the $n \times n \times n$ system, in section 3 we arrive at the necessary and sufficient criterion for the interconvertibility between all-tripartite Gaussian states. Based on the mathematically abstract criterion established there, we will further propose in section 4 a convenient method to determine whether a given transformation is possible or not. Finally, we conclude this paper in section 5.

2. The normal form

An $n \times n \times n$ continuous tripartite quantum system can be conveniently described via its Wigner phase-space distribution functions. The canonical coordinate–momentum variables for the all-tripartite system $\left(q_i^\alpha, p_i^\alpha; i = 1, \ldots, n; \alpha = \mathrm{A, B, C}\right)$ can be combined in a $6n$-dimensional vector, $Q = (Q_1, Q_2, \ldots, Q_{6n})$, where $Q_1 \equiv q_1^{\mathrm{A}}, Q_2 \equiv p_1^{\mathrm{A}}$, and so on. Let $J \equiv \{J_{ab}; a, b = 1, \ldots, 6n\}$ be the $6n \times 6n$ matrix of canonical commutators, $J_{ab} \equiv -\mathrm{i}[Q_a, Q_b]$; i.e.,

$$J = \bigoplus_{j=1}^{3n} J_1, \qquad J_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

Denote $\Gamma \equiv \{\Gamma_{ab}; a, b = 1, \ldots, 6n\}$ as the covariance matrix for the tripartite system correlations, $\Gamma_{ab} = \langle \delta Q_a \delta Q_b + \delta Q_b \delta Q_a \rangle$, where $\delta Q_a \equiv Q_a - \langle Q_a \rangle$, $\langle O \rangle \equiv \mathrm{tr}(O\rho)$ and ρ is the density matrix of the all-tripartite system. The $6n \times 6n$ matrix Γ that is real and symmetric can be written in the block form

$$\Gamma = \begin{pmatrix} \Gamma_{\mathrm{AA}} & \Gamma_{\mathrm{AB}} & \Gamma_{\mathrm{AC}} \\ \Gamma_{\mathrm{BA}} & \Gamma_{\mathrm{BB}} & \Gamma_{\mathrm{BC}} \\ \Gamma_{\mathrm{CA}} & \Gamma_{\mathrm{CB}} & \Gamma_{\mathrm{CC}} \end{pmatrix}, \tag{2}$$

in terms of Alice, Bob and Charlie's locations and their correlations. Here, each $\Gamma_{\alpha\beta}$, with $\alpha, \beta = \mathrm{A, B}$, or C, is a $2n \times 2n$ matrix. To study the entanglement transformation under local quantum operations for Gaussian states (LOG), we shall only consider a set of locally nonequivalent Gaussian states [3]. It is well known that any two equivalent Gaussian states can be converted into each other by means of local symplectic transformations, forming the group of real symplectic transformations $Sp(6n, \mathbf{R})$ [3]. Consequently, the entanglement property of a Gaussian state ρ is completely characterized by its real and symmetric $6n \times 6n$ correlation matrix Γ. We shall hereafter use Γ, without explicitly referring to ρ, to discuss the entanglement transformation under LOG. Furthermore, a physically legitimate state should satisfy the positivity, which in terms of covariance matrix Γ is the uncertainty relation

$$\Gamma - \mathrm{i}J \geqslant 0. \tag{3}$$

[3] It was found that [7] only gave a criterion for the special three-mode tripartite Gaussian states. We wish to present the general criterion for this case in errata in future.

The equal sign holds iff (if and only if) the Gaussian state is pure. Thus, the entanglement transformation under LOG constitutes actually a set of *completely positive definite* maps of Gaussians [9]. Obviously, each of the diagonal blocks $\Gamma_{\alpha\alpha}$, α = A, B, C in equation (2), is positive definite and can be diagonalized by a symplectic transformation [3]. We can choose a symplectic transformation $S_\alpha \in Sp(2n, \mathbf{R})$, such that

$$S_\alpha^T \Gamma_{\alpha\alpha} S_\alpha = \bigoplus_{j=1}^{n} D_j^\alpha, \tag{4}$$

with $D_j^\alpha \geqslant I_2$ being a 2×2 diagonal matrix of two equal diagonal elements [3]. Here, I_2 denotes the 2×2 identity matrix. Therefore, $S = \bigoplus_\alpha S_\alpha \in Sp(6n, \mathbf{R})$ diagonalizes simultaneously all $\Gamma_{\alpha\alpha}$, α = A, B, C in equation (2), into the form of $\{D_j^\alpha\}$ with the property just described, which is *invariance under a symplectic and orthogonal transformation*. We now introduce a theorem in the quest of further simplification for at least two of the three off-diagonal block matrices, $\Gamma_{\alpha\beta}$ with $\alpha < \beta$ (i.e., Γ_{AB}, Γ_{AC} and Γ_{BC}) in Γ in equation (2).

Theorem (the normal form of $n \times n \times n$ Gaussian state)**.** *Any all-tripartite Gaussian state specified by its covariance matrix* Γ *can be transformed into the state with a covariance matrix* Λ *of the normal form as follows:*

$$\Lambda = \begin{pmatrix} \Lambda_{\mathrm{AA}} & \Lambda_{\mathrm{AB}} & \Lambda_{\mathrm{AC}} \\ \Lambda_{\mathrm{BA}} & \Lambda_{\mathrm{BB}} & \Lambda_{\mathrm{BC}} \\ \Lambda_{\mathrm{CA}} & \Lambda_{\mathrm{CB}} & \Lambda_{\mathrm{CC}} \end{pmatrix}, \tag{5}$$

with

$$\Lambda_{\mathrm{AA}} = \bigoplus_{\mu=1}^{n} D_\mu, \qquad \Lambda_{\mathrm{BB}} = \bigoplus_{\mu=n+1}^{2n} D_\mu, \qquad \Lambda_{\mathrm{CC}} = \bigoplus_{\mu=2n+1}^{3n} D_\mu,$$

$$\Lambda_{\mathrm{AB}} = \Lambda_{\mathrm{BA}}^T = \begin{pmatrix} U_{11} & \cdots & U_{1,n} \\ \cdots & \cdots & \cdots \\ U_{n,1} & \cdots & U_{n,n} \end{pmatrix},$$

$$\Lambda_{\mathrm{BC}} = \Lambda_{\mathrm{CB}}^T = \begin{pmatrix} V_{11} & \cdots & V_{1,n} \\ \cdots & \cdots & \cdots \\ V_{n,1} & \cdots & V_{n,n} \end{pmatrix},$$

$$\Lambda_{\mathrm{AC}} = \Lambda_{\mathrm{CA}}^T = \begin{pmatrix} P_{11} & \cdots & P_{1,n} \\ \cdots & \cdots & \cdots \\ P_{n,1} & \cdots & P_{n,n} \end{pmatrix},$$

where both $D_\mu \propto I_2$ *and* U_{ii}, $i = 1, \ldots, n$, *are diagonal* 2×2 *matrices and* V_{ii} *are upper-triangular* 2×2 *matrices. The others are in general off-diagonal* 2×2 *matrices.*

Proof. By virtue of [3], we can always find such a set of symplectic and orthogonal maps on individual modes of local phase spaces, i.e., $O_j^\alpha \in SO_2$, with α = A, B, C and $j = 1, \ldots, n$, to be of the form

$$O_j^\alpha = \begin{pmatrix} \cos\theta_j^\alpha/2 & \sin\theta_j^\alpha/2 \\ -\sin\theta_j^\alpha/2 & \cos\theta_j^\alpha/2 \end{pmatrix} = \exp\left(J_1 \theta_j^\alpha\right). \tag{6}$$

Here, $-\pi < \theta_j^\alpha \leqslant \pi$ and J_1 is given by the second identity in equation (1). Obviously, O_j^α does not alter $D_j^\alpha \propto I_2$ in equation (4). We now consider a symplectic and orthogonal map consisting of the following two steps. The first step is $S = \bigoplus_\alpha S_\alpha \in Sp(6n, \mathbf{R})$ by which $\Gamma_{\alpha\beta} \longrightarrow \Gamma'_{\alpha\beta} = S_\alpha^T \Gamma_{\alpha\beta} S_\beta$, with the diagonal ($\alpha = \beta$) blocks being diagonalized into the form of equation (4). The second step is the map of $O = \bigoplus_\alpha \bigoplus_{j=1}^n O_j^\alpha \in SO(6n, \mathbf{R})$ which thus diagonalizes all U_{ii} and triangularizes V_{ii} but keeps the others off-diagonal in general. We have thus arrived at the statement of the theorem. □

For the sake of convenience, we shall hereafter denote

$$\begin{aligned} &D_\mu \equiv \xi_\mu I_2, && \mu = 1, \ldots, 3n, \\ &\Lambda_{\rm AB} \equiv (U_{ij}) \equiv \Lambda_{\rm AB}(\eta_\nu), && \nu = 1, \ldots, 4n^2 - 2n, \\ &\Lambda_{\rm BC} \equiv (V_{ij}) \equiv \Lambda_{\rm BC}(\eta_\nu), && \nu = 4n^2 - 2n + 1, \ldots, 8n^2 - 3n, \\ &\Lambda_{\rm AC} \equiv (P_{ij}) \equiv \Lambda_{\rm AC}(\eta_\nu), && \nu = 8n^2 - 3n + 1, \ldots, 12n^2 - 3n. \end{aligned}$$

The normal form of an all-tripartite $(n \times n \times n)$ Gaussian state is therefore characterized by $\{\xi_\mu, \eta_\nu; \mu = 1, \ldots, 3n, \nu = 1, \ldots, 12n^2 - 3n\}$. These $12n^2$ independent parameters are restricted by the uncertainty relation $\Lambda - {\rm i}J \geqslant 0$, which can be recognized via equation (3) and the fact that the symplectic matrix J in equation (1) is invariant under the prescribed normal form transformation (cf equations (4) and (5)).

3. The criterion in abstract manner

By means of the theorem, we may construct a $12n^2$ dimensional vector space $\mathbf{R}^{12n^2}$ with its elements $(\xi_1, \ldots, \xi_{3n}, \eta_1, \ldots, \eta_{12n^2-3n})$, which will also be abbreviated as (ξ_μ, η_ν) whenever it causes no confusion. Thus, an orbit $O(\Lambda)$ of Λ can be completely characterized by the vector $(\xi_\mu, \eta_\nu) \in \mathbf{R}^{12n^2}$.

It has been established that there are a set of functions, called minimal functions, playing crucial roles in determining whether it is possible to transform one state Λ into another Λ'; i.e., the criteria of interconvertibility for any two mixed Gaussian states [5, 7, 8]. Let the matrix function $H : \mathbf{R}^{2n\times 2n}_{L_{\alpha\beta}} \to \mathbf{R}^{2n\times 2n}$, defined as

$$H(P, V, F) := P - F(V + {\rm i}J)F^T + {\rm i}J, \tag{7}$$

where $\mathbf{R}^{2n\times 2n}$ is a $2n \times 2n$ matrix space, the subscript $L_{\alpha\beta} (\alpha \neq \beta)$ denotes the restriction of the subspace $\mathbf{R}^{2n\times 2n}_{L_{\alpha\beta}} (\alpha \neq \beta)$ resulting from the uncertainty relation in equation (3), while H, P, V and F are $2n \times 2n$ matrices and $J = \oplus_{k=1}^n J_1$ (equation (1)). For a pair of all-tripartite Gaussian states (Λ, Λ') with corresponding submatrices $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha\beta}$, we define three independent functions, $H^{\rm I(\Lambda\to\Lambda')}$, $H^{\rm II(\Lambda\to\Lambda')}$ or $H^{\rm II'(\Lambda\to\Lambda')}$, and $H^{\rm III(\Lambda\to\Lambda')} : \mathbf{R}^{2n\times 2n}_{L_{\alpha\beta}} \to \mathbf{R}^{2n\times 2n}$ by virtue of equation (7), respectively, as

$$H^{\rm I(\Lambda\to\Lambda')}(X) := H\big(\Lambda'_{\rm AA}, \Lambda_{\rm AA}, X\Lambda_{\rm AB}^{-1}\big), \tag{8a}$$

$$H^{\rm II(\Lambda\to\Lambda')}(X) := H(\Lambda'_{\rm BB}, \Lambda_{\rm BB}, \Lambda'_{\rm AB}X^{-1}), \tag{8b}$$

$$H^{\rm II'(\Lambda\to\Lambda')}(Y) := H\big(\Lambda'_{\rm BB}, \Lambda_{\rm BB}, Y\Lambda_{\rm BC}^{-1}\big), \tag{8c}$$

$$H^{\rm III(\Lambda\to\Lambda')}(Y) := H(\Lambda'_{\rm CC}, \Lambda_{\rm CC}, \Lambda'_{\rm BC}Y^{-1}). \tag{8d}$$

The above three independent functions (noting that equations (8*b*) and (8*c*) are two equivalent definitions) constitute the set of minimal functions. Making use of the normal form Λ and the above definitions, we can arrive at four propositions as follows.

Proposition 1 (restricted to Alice's location). *Let Λ and Λ^{I} be Gaussian states of an $n\times n\times n$ system, where*

$$\Lambda=\begin{pmatrix}\Lambda_{\mathrm{AA}} & \Lambda_{\mathrm{AB}} & \Lambda_{\mathrm{AC}}\\ \Lambda_{\mathrm{BA}} & \Lambda_{\mathrm{BB}} & \Lambda_{\mathrm{BC}}\\ \Lambda_{\mathrm{CA}} & \Lambda_{\mathrm{CB}} & \Lambda_{\mathrm{CC}}\end{pmatrix},\qquad \Lambda^{\mathrm{I}}=\begin{pmatrix}\Lambda'_{\mathrm{AA}} & \Lambda^{\mathrm{I}}_{\mathrm{AB}} & \Lambda^{\mathrm{I}}_{\mathrm{AC}}\\ \Lambda^{\mathrm{I}}_{\mathrm{BA}} & \Lambda_{\mathrm{BB}} & \Lambda_{\mathrm{BC}}\\ \Lambda^{\mathrm{I}}_{\mathrm{CA}} & \Lambda_{\mathrm{CB}} & \Lambda_{\mathrm{CC}}\end{pmatrix}.$$

Then $\Lambda\rightarrow\Lambda^{\mathrm{I}}$ under LOG at Alice's location iff

$$H_s^{\mathrm{I}(\Lambda\rightarrow\Lambda^{\mathrm{I}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)\in S_{\mathrm{I}}^{(\Lambda\rightarrow\Lambda^{\mathrm{I}})} \tag{9}$$

and

$$\Lambda^{\mathrm{I}}_{\mathrm{AB}}\Lambda^{-1}_{\mathrm{AB}}=\Lambda^{\mathrm{I}}_{\mathrm{AC}}\Lambda^{-1}_{\mathrm{AC}}, \tag{10}$$

where

$$S_{\mathrm{I}}^{(\Lambda\rightarrow\Lambda^{\mathrm{I}})}:=\left\{H_s^{\mathrm{I}(\Lambda\rightarrow\Lambda^{\mathrm{I}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)\middle|\left|H_s^{\mathrm{I}(\Lambda\rightarrow\Lambda^{\mathrm{I}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)\right|\geqslant 0\right\}, \tag{11}$$

with $\left|H_s^{\mathrm{I}(\Lambda\rightarrow\Lambda^{\mathrm{I}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)\right|$ being the leading principal minor of $H^{\mathrm{I}(\Lambda\rightarrow\Lambda^{\mathrm{I}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)$, and $s=1,\ldots,2n$ [10].

Proposition 2 (restricted to Bob's location). *Let Λ^{I} and Λ^{II} be Gaussian states of an $n\times n\times n$ system, where*

$$\Lambda^{\mathrm{I}}=\begin{pmatrix}\Lambda'_{\mathrm{AA}} & \Lambda^{\mathrm{I}}_{\mathrm{AB}} & \Lambda^{\mathrm{I}}_{\mathrm{AC}}\\ \Lambda^{\mathrm{I}}_{\mathrm{BA}} & \Lambda_{\mathrm{BB}} & \Lambda_{\mathrm{BC}}\\ \Lambda^{\mathrm{I}}_{\mathrm{CA}} & \Lambda_{\mathrm{CB}} & \Lambda_{\mathrm{CC}}\end{pmatrix},\qquad \Lambda^{\mathrm{II}}=\begin{pmatrix}\Lambda'_{\mathrm{AA}} & \Lambda'_{\mathrm{AB}} & \Lambda^{\mathrm{I}}_{\mathrm{AC}}\\ \Lambda'_{\mathrm{BA}} & \Lambda'_{\mathrm{BB}} & \Lambda^{\mathrm{I}}_{\mathrm{CC}}\\ \Lambda^{\mathrm{I}}_{\mathrm{CA}} & \Lambda^{\mathrm{I}}_{\mathrm{CB}} & \Lambda_{\mathrm{CC}}\end{pmatrix}.$$

Then $\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}}$ under LOG at Bob's location iff

$$H_s^{\mathrm{II}(\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)\in S_{\mathrm{II}}^{(\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}})} \tag{12a}$$

or

$$H_s^{\mathrm{II}'(\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}})}\left(\Lambda^{\mathrm{I}}_{\mathrm{BC}}\right)\in S_{\mathrm{II}'}^{(\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}})}, \tag{12b}$$

and

$$\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)^{-1}\Lambda^{\mathrm{II}}_{\mathrm{AB}}=\left(\Lambda^{\mathrm{I}}_{\mathrm{BC}}\right)^{-1}\Lambda^{\mathrm{II}}_{\mathrm{BC}}, \tag{13}$$

where $S_{\mathrm{II}}^{(\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}})}$ and $S_{\mathrm{II}'}^{(\Lambda^{\mathrm{I}}\rightarrow\Lambda^{\mathrm{II}})}$ are defined similarly as in equation (11) with their individual functions, respectively.

Proposition 3 (restricted to Charlie's location). *Let Λ^{II} and Λ' be Gaussian states of an $n\times n\times n$ system, where*

$$\Lambda^{\mathrm{II}}=\begin{pmatrix}\Lambda'_{\mathrm{AA}} & \Lambda'_{\mathrm{AB}} & \Lambda^{\mathrm{I}}_{\mathrm{AC}}\\ \Lambda'_{\mathrm{BA}} & \Lambda'_{\mathrm{BB}} & \Lambda^{\mathrm{I}}_{\mathrm{BC}}\\ \Lambda^{\mathrm{I}}_{\mathrm{CA}} & \Lambda^{\mathrm{I}}_{\mathrm{CB}} & \Lambda_{\mathrm{CC}}\end{pmatrix},\qquad \Lambda'=\begin{pmatrix}\Lambda'_{\mathrm{AA}} & \Lambda'_{\mathrm{AB}} & \Lambda'_{\mathrm{AC}}\\ \Lambda'_{\mathrm{BA}} & \Lambda'_{\mathrm{BB}} & \Lambda'_{\mathrm{BC}}\\ \Lambda'_{\mathrm{CA}} & \Lambda'_{\mathrm{CB}} & \Lambda'_{\mathrm{CC}}\end{pmatrix}.$$

Then $\Lambda^{\mathrm{II}}\rightarrow\Lambda'$ under LOG at Charlie's location iff

$$H_s^{\mathrm{III}(\Lambda^{\mathrm{II}}\rightarrow\Lambda')}\left(\Lambda^{\mathrm{I}}_{\mathrm{BC}}\right)\in S_{\mathrm{III}}^{(\Lambda^{\mathrm{II}}\rightarrow\Lambda')} \tag{14}$$

and

$$\left(\Lambda^{\mathrm{II}}_{\mathrm{BC}}\right)^{-1}\Lambda'_{\mathrm{BC}}=\left(\Lambda^{\mathrm{II}}_{\mathrm{AC}}\right)^{-1}\Lambda'_{\mathrm{AC}}, \tag{15}$$

where $S_{\mathrm{III}}^{(\Lambda^{\mathrm{II}}\rightarrow\Lambda')}$ is defined similarly as in equation (11).

Proposition 4 ($\Lambda \to \Lambda'$ under LOG). *Let Λ and Λ' be Gaussian states of an $n \times n \times n$ system with $2n \times 2n$ submatrices $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha\beta}$, respectively, where $\alpha, \beta = \mathrm{A},\ \mathrm{B},\ \mathrm{C}$. Then $\Lambda \to \Lambda'$ under LOG iff there exist two of the matrices, $X, Y \in \mathbf{R}_L^{2n\times 2n}$, such that ($\Phi$ is an empty set)*

$$S_1^{(\Lambda\to\Lambda')} := S_{\mathrm{I}}^{(\Lambda\to\Lambda^{\mathrm{I}})} \cap S_{\mathrm{II}}^{(\Lambda^{\mathrm{I}}\to\Lambda^{\mathrm{II}})} \neq \Phi, \tag{16}$$

$$S_2^{(\Lambda\to\Lambda')} := S_{\mathrm{II}'}^{(\Lambda^{\mathrm{I}}\to\Lambda^{\mathrm{II}})} \cap S_{\mathrm{III}}^{(\Lambda^{\mathrm{II}}\to\Lambda')} \neq \Phi, \tag{17}$$

together with $\Lambda^{\mathrm{I}}_{\mathrm{AB}}\Lambda^{-1}_{\mathrm{AB}} = \Lambda^{\mathrm{I}}_{\mathrm{AC}}\Lambda^{-1}_{\mathrm{AC}}$, $\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)^{-1}\Lambda^{\mathrm{II}}_{\mathrm{AB}} = \left(\Lambda^{\mathrm{I}}_{\mathrm{BC}}\right)^{-1}\Lambda^{\mathrm{II}}_{\mathrm{BC}}$, and $\left(\Lambda^{\mathrm{II}}_{\mathrm{BC}}\right)^{-1}\Lambda'_{\mathrm{BC}} = \left(\Lambda^{\mathrm{II}}_{\mathrm{AC}}\right)^{-1}\Lambda'_{\mathrm{AC}}$ (equations (10), (13) and (15)).

Proof of proposition 1. This proposition tells about what conditions must be satisfied when a LOG is applied on all n modes merely in system A and two symplectic operations in systems B and C. If we want to implement the LOG, such a map has to be Gaussian completely positive [9], which is reflected on the covariance matrix level as a map of

$$\Lambda \mapsto M^T \Lambda M + G, \tag{18}$$

where M and G are real $6n \times 6n$ matrices and G is also symmetric. The matrix inequality

$$G + \mathrm{i}J - \mathrm{i}M^T J M \geqslant 0 \tag{19}$$

incorporates the complete positivity of the map. For the case of proposition 1, the completely positive map (18) can be reduced as

$$\Lambda^{\mathrm{I}} = (M^{\mathrm{I}})^T \Lambda M^{\mathrm{I}} + G^{\mathrm{I}}. \tag{20}$$

This represents a LOG restricted to Alice's location and symplectic operations to Bob and Charlie's locations, where either Λ or Λ^{I} is already in the normal form, i.e., $\Lambda_{\alpha\alpha}\left(\Lambda^{\mathrm{I}}_{\alpha\alpha}\right)$ and $\Lambda_{\alpha\beta}\left(\Lambda^{\mathrm{I}}_{\alpha\beta}\right)$ ($\alpha \neq \beta$) are chosen according to the theorem, corresponding to respective submatrices. We can directly choose $M^{\mathrm{I}} = \oplus_{\alpha=1}^{3} M^{\mathrm{I}}_{\alpha}$ and $G^{\mathrm{I}} = \oplus_{\alpha=1}^{3} G^{\mathrm{I}}_{\alpha}$, where $M^{\mathrm{I}}_{\mathrm{B}}, M^{\mathrm{I}}_{\mathrm{C}} \in Sp(2n, \mathbf{R})$, $G^{\mathrm{I}}_{\mathrm{B}} = G^{\mathrm{I}}_{\mathrm{C}} = \mathbf{0}$ and $G^{\mathrm{I}}_{\mathrm{A}}$ is symmetric. From equation (20), we can easily derive the following six reduced equations:

$$\Lambda_{\mathrm{BB}} = \left(M^{\mathrm{I}}_{\mathrm{B}}\right)^T \Lambda_{\mathrm{BB}} M^{\mathrm{I}}_{\mathrm{B}} \tag{21a}$$

$$\Lambda_{\mathrm{BC}} = \left(M^{\mathrm{I}}_{\mathrm{B}}\right)^T \Lambda_{\mathrm{BC}} M^{\mathrm{I}}_{\mathrm{C}} \tag{21b}$$

$$\Lambda_{\mathrm{CC}} = \left(M^{\mathrm{I}}_{\mathrm{C}}\right)^T \Lambda_{\mathrm{CC}} M^{\mathrm{I}}_{\mathrm{C}} \tag{21c}$$

and

$$\Lambda'_{\mathrm{AA}} = \left(M^{\mathrm{I}}_{\mathrm{A}}\right)^T \Lambda_{\mathrm{AA}} M^{\mathrm{I}}_{\mathrm{A}} + G^{\mathrm{I}}_{\mathrm{A}} \tag{22a}$$

$$\Lambda^{\mathrm{I}}_{\mathrm{AB}} = \left(M^{\mathrm{I}}_{\mathrm{A}}\right)^T \Lambda_{\mathrm{AB}} M^{\mathrm{I}}_{\mathrm{B}} \tag{22b}$$

$$\Lambda^{\mathrm{I}}_{\mathrm{AC}} = \left(M^{\mathrm{I}}_{\mathrm{A}}\right)^T \Lambda_{\mathrm{AC}} M^{\mathrm{I}}_{\mathrm{C}}. \tag{22c}$$

Equations (21) demand that $M^{\mathrm{I}}_{\mathrm{B}}$ and $M^{\mathrm{I}}_{\mathrm{C}}$ be both symplectic and orthogonal matrices. Hence, we can always choose

$$M^{\mathrm{I}}_{\mathrm{B}} = \bigoplus_{i=1}^{n} O_i, \qquad M^{\mathrm{I}}_{\mathrm{C}} = \bigoplus_{i=n+1}^{2n} O_i,$$

where O_i is given by equation (6). The matrix inequality (19) amounts to the reduced matrix inequality

$$H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})} : G_{\mathrm{A}}^{\mathrm{I}} + \mathrm{i}J_n - \mathrm{i}\big(M_{\mathrm{A}}^{\mathrm{I}}\big)^T J_n M_{\mathrm{A}}^{\mathrm{I}} \geqslant 0. \tag{23}$$

It is now very apparent that $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})}$ is a $2n \times 2n$ Hermitian positive definite matrix and $J_n = \oplus_{k=1}^{n} J_1$. It had been proved in [8] that det $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})}$ is maximal, i.e., the optimal result can be obtained if we choose all $\theta_i = 0$, i.e., $M_{\mathrm{B}}^{\mathrm{I}} = M_{\mathrm{C}}^{\mathrm{I}} = \oplus_{i=1}^{n} O_i = \oplus_{i=n+1}^{2n} O_i = \oplus_{i=1}^{n} I_2 = I_{(2n)}$. Here, $I_{(2n)}$ denotes a $2n \times 2n$ identity matrix. Thus we have the results:

$$M_{\mathrm{A}}^{\mathrm{I}} = \big(\Lambda_{\mathrm{AB}}^{\mathrm{I}}\Lambda_{\mathrm{AB}}^{-1}\big)^T \tag{24}$$

and

$$G_{\mathrm{A}}^{\mathrm{I}} = \Lambda'_{\mathrm{AA}} - \Lambda_{\mathrm{AB}}^{\mathrm{I}}\Lambda_{\mathrm{AB}}^{-1}\Lambda_{\mathrm{AA}}\big(\Lambda_{\mathrm{AB}}^{-1}\big)^T\big(\Lambda_{\mathrm{AB}}^{\mathrm{I}}\big)^T. \tag{25}$$

If we bring the results of equations (24) and (25) into definition (8*a*), then $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})}\big(\Lambda_{\mathrm{AB}}^{\mathrm{I}}\big)$ i.e., the minimal function for the case of proposition 1, can be obtained. By means of the theorem that Hermitian matrix is positive definite iff all leading principal minors of $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})}$ are positive, the inequality condition (23) amounts to condition (9). With regard to equality of equation (10), we can easily obtain it from equations (22*b*) and (22*c*). We thus complete the proof of proposition 1. □

Proof of propositions 2 and 3. It is very easy for one to do it by a method analogous to the proof of proposition 1. □

Proof of proposition 4 ($\Lambda \to \Lambda'$ under LOG)**.** In order to complete the proof, we need to break a whole transformation $\Lambda \to \Lambda'$ into three partial transformations: $\Lambda \to \Lambda^{\mathrm{I}}$, $\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}}$ and $\Lambda^{\mathrm{II}} \to \Lambda'$; each of them is restricted in one system as a LOG and in the other two systems as symplectic operations (cf figure 1). Set Λ^{I} and Λ^{II} as the two covariance matrices that represent two intermediate states,

$$\Lambda^{\mathrm{I}} = \begin{pmatrix} \Lambda'_{\mathrm{AA}} & X & \Lambda_{\mathrm{AC}}^{\mathrm{I}}(X) \\ X & \Lambda_{\mathrm{BB}} & \Lambda_{\mathrm{BC}} \\ \Lambda_{\mathrm{CA}}^{\mathrm{I}}(X) & \Lambda_{\mathrm{CB}} & \Lambda_{\mathrm{CC}} \end{pmatrix}, \tag{26}$$

and

$$\Lambda^{\mathrm{II}} = \begin{pmatrix} \Lambda'_{\mathrm{AA}} & \Lambda'_{\mathrm{AB}} & \Lambda_{\mathrm{AC}}^{\mathrm{I}}(X) \\ \Lambda'_{\mathrm{BA}} & \Lambda'_{\mathrm{BB}} & \Lambda_{\mathrm{BC}}^{\mathrm{I}} \\ \Lambda_{\mathrm{CA}}^{\mathrm{I}}(X) & \Lambda_{\mathrm{CB}}^{\mathrm{I}} & \Lambda_{\mathrm{CC}} \end{pmatrix} \tag{27}$$

or

$$\Lambda^{\mathrm{II}'} = \begin{pmatrix} \Lambda'_{\mathrm{AA}} & \Lambda'_{\mathrm{AB}} & \Lambda_{\mathrm{AC}}^{\mathrm{I}}(Y) \\ \Lambda'_{\mathrm{BA}} & \Lambda'_{\mathrm{BB}} & Y \\ \Lambda_{\mathrm{CA}}^{\mathrm{I}}(Y) & Y & \Lambda_{\mathrm{CC}} \end{pmatrix}, \tag{28}$$

where $\Lambda_{\mathrm{AC}}^{\mathrm{I}}(X)\big(\Lambda_{\mathrm{CA}}^{\mathrm{I}}(X)\big)$ or $\Lambda_{\mathrm{AC}}^{\mathrm{I}}(Y)\big(\Lambda_{\mathrm{CA}}^{\mathrm{I}}(Y)\big)$ can be obtained by equality conditions (10) and (15). Firstly, according to propositions 1 and 2, both $\Lambda \to \Lambda^{\mathrm{I}}$ and $\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}}$ under LOG iff equations (9), (10), (12*a*) or (12*b*) and (13) hold simultaneously. This means that iff matrix $X \in \mathbf{R}_L^{2n\times 2n}$ exists such that

$$S_1^{(\Lambda\to\Lambda')} := S_{\mathrm{I}}^{(\Lambda\to\Lambda^{\mathrm{I}})} \cap S_{\mathrm{II}}^{(\Lambda^{\mathrm{I}}\to\Lambda^{\mathrm{II}})} \neq \Phi, \tag{29}$$

together with $\Lambda_{\mathrm{AB}}^{\mathrm{I}}\Lambda_{\mathrm{AB}}^{-1} = \Lambda_{\mathrm{AC}}^{\mathrm{I}}\Lambda_{\mathrm{AC}}^{-1}$ and $\big(\Lambda_{\mathrm{AB}}^{\mathrm{I}}\big)^{-1}\Lambda_{\mathrm{AB}}^{\mathrm{II}} = \big(\Lambda_{\mathrm{BC}}^{\mathrm{I}}\big)^{-1}\Lambda_{\mathrm{BC}}^{\mathrm{II}}$ (equations (10) and (13)), then both $\Lambda \to \Lambda^{\mathrm{I}}$ and $\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}}$ under LOG. Finally, by virtue of propositions 2

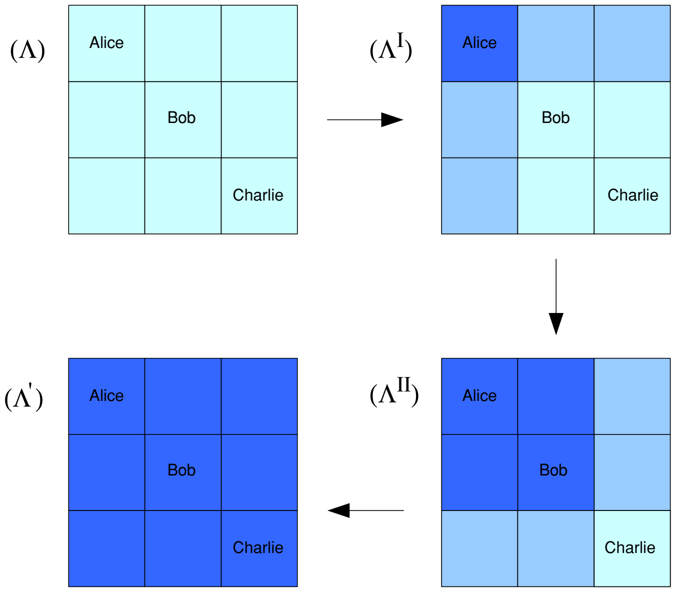

Figure 1. The square blocks on the diagonal labelled as Alice, Bob and Charlie, respectively, represent three parties in quantum communication while the other non-diagonal square blocks just denote their correlativity. The transformation from Λ to Λ' can be decomposed into three steps as $\Lambda \to \Lambda^{\mathrm{I}}$, $\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}}$ and $\Lambda^{\mathrm{II}} \to \Lambda'$, which are restricted in one system as LOG and in the other two as symplectic operations, respectively.

and 3, we conclude in the same manner as above that $\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}}$ and $\Lambda^{\mathrm{II}} \to \Lambda'$ under LOG iff matrix $Y \in \mathbf{R}_L^{2n\times 2n}$ exists so that

$$S_2^{(\Lambda\to\Lambda')} := S_{\mathrm{II}'}^{(\Lambda^{\mathrm{I}}\to\Lambda^{\mathrm{II}})} \cap S_{\mathrm{III}}^{(\Lambda^{\mathrm{II}}\to\Lambda')} \neq \Phi, \tag{30}$$

together with $\left(\Lambda_{\mathrm{AB}}^{\mathrm{I}}\right)^{-1}\Lambda_{\mathrm{AB}}^{\mathrm{II}} = \left(\Lambda_{\mathrm{BC}}^{\mathrm{I}}\right)^{-1}\Lambda_{\mathrm{BC}}^{\mathrm{II}}$ and $\left(\Lambda_{\mathrm{BC}}^{\mathrm{II}}\right)^{-1}\Lambda'_{\mathrm{BC}} = \left(\Lambda_{\mathrm{AC}}^{\mathrm{II}}\right)^{-1}\Lambda'_{\mathrm{AC}}$ (equations (13) and (15)). Then we arrive at the content of proposition 4. □

In view of proposition 4, it is clear that the conditions of convertibility for all-tripartite Gaussian states are much stricter than for any bipartite cases as here equality restrictions arise. This new type of restriction on the convertibility does not appear in bipartite cases as it is not a necessity for complete positivity of LOG but a speciality of tripartite cases.

4. Necessary conditions in an efficient manner[4]

Presented thus far is a necessary and sufficient criterion for the interconvertibility between all-tripartite Gaussian states. However, due to its abstractivity in mathematics, it is yet to provide an operable method for practically checking whether given $n \times n \times n$ Gaussian states are interconvertible or not. In fact, proposition 4 practically transforms a complex problem, a $6n \times 6n$ Hermitian matrix, into a relatively simple set of $2n \times 2n$ matrices, so that the positivity of the original $6n \times 6n$ matrix is obtained in terms of a set of matrix inequalities

[4] All of our proofs of the propositions in section 4 are based on the fact that another of our proofs (in [8]) is complete. Now we know that it is not complete though the corresponding proposition is very likely to be real. This means that we cannot be sure that the method proposed in section 4 is necessarily applicable. We hope that somebody can give the full proof for note (11) in [8] in the future.

and equalities. In the following, we shall present an efficient and more practical criterion in terms of polynomial inequalities that can be inferred from proposition 4. To achieve a polynomial-based operable criterion, the $12n^2$-dimensional parameter (ξ_μ, η_ν)-vector space would rather be adopted than the original all-tripartite covariance matrix space. We shall also determine whether a block Hermitian matrix is positive by using the following lemma.

Lemma 2. *If a $2n \times 2n$ matrix $H \geqslant 0$, then both $H_{\mu\mu} \geqslant 0$ and $\operatorname{tr} H_{\mu\mu} \geqslant 0$, where H is Hermitian and $H_{\mu\mu}$ are 2×2 blocks on the diagonal of H.*

Proof. See [8], proof of proposition 1. □

Defining a function

$$F_\alpha(a_\alpha, b_\alpha, c_{\alpha\beta}) := (a_\alpha)^2 - a_\alpha \sum_\beta b_\beta \|c_{\alpha\beta}\|_{\mathrm{F}}^2 + \left|\sum_\beta b_\beta c_{\alpha\beta} c_{\beta\alpha}\right| - \left(1 - 2\sum_\beta |c_{\alpha\beta}| + \sum_{\beta,\gamma} |c_{\alpha\beta}||c_{\gamma\alpha}|\right),$$

where

$$c_{\alpha\beta} = c_{\alpha\beta}(\overline{\eta}^\lambda \eta^\rho) = \begin{cases} \sum_{l=1}^n \overline{U}_{l\alpha}^T U'^T_{\beta l} \\ \sum_{l=1}^n \overline{U}_{(\alpha-n)l} U'_{l(\beta-n)} \\ \left(\text{or} \sum_{l=1}^n \overline{V}_{l(\alpha-n)}^T V'^T_{(\beta-n)l}\right) \\ \sum_{l=1}^n \overline{V}_{(\alpha-2n)l} V'_{l(\beta-2n)} \end{cases}$$

with $\alpha, \beta = 1, \ldots, n, n+1, \ldots, 2n$ or $2n+1, \ldots, 3n$, respectively, for the above three cases. L_α denotes the restriction to the corresponding subspace $\mathbf{R}_{L_\alpha}$ resulting from the uncertainty relation. $\overline{U}_{l\alpha}(\overline{V}_{l\alpha})$ are the 2×2 matrix blocks of the inverse of $U(V)$, and $\overline{\eta}^\lambda$ are the elements of $U(V)^{-1}$ (corresponding to η^λ), and $\|\,\|_{\mathrm{F}}$ denotes the Frobenius norm. For a pair (Λ, Λ') of all-tripartite covariance matrices with their corresponding vectors $(\xi_\alpha, \eta_\lambda)$ and $(\xi'_\alpha, \eta'_\lambda)$, where $\alpha = 1, \ldots, 3n$, and $\lambda = 1, \ldots, (12n^2 - 3n)$, we define four families of minimal functions, $h_\mu^{\mathrm{I}(\Lambda\to\Lambda')}$, $h_\nu^{\mathrm{II}(\Lambda\to\Lambda')}$ or $h_\nu^{\mathrm{II}'(\Lambda\to\Lambda')}$, $h_\sigma^{\mathrm{III}(\Lambda\to\Lambda')}$:

$$\mathbf{R}_L^1 \times \mathbf{R}_L^2 \times \cdots \times \mathbf{R}_L^{(4n^2-2n)} \to \mathbf{R}$$

or

$$\mathbf{R}_L^{(4n^2-2n+1)} \times \mathbf{R}_L^{(4n^2-2n+2)} \times \cdots \times \mathbf{R}_L^{(8n^2-3n)} \to \mathbf{R}$$

as

$$h_\mu^{\mathrm{I}(\Lambda\to\Lambda')}(x_\lambda) := H_\mu^{\mathrm{I}}[\xi'_\mu, \xi_\alpha, c_{\mu\alpha}(x_\lambda)], \tag{31a}$$

$$h_\nu^{\mathrm{II}(\Lambda\to\Lambda')}(x_\lambda) := H_\nu^{\mathrm{II}}[\xi'_\nu, \xi_\beta, c_{\nu\beta}(x_\lambda)], \tag{31b}$$

$$h_\nu^{\mathrm{II}'(\Lambda\to\Lambda')}(y_\rho) := H_\nu^{\mathrm{II}'}[\xi'_\nu, \xi_\beta, c_{\nu\beta}(y_\rho)], \tag{31c}$$

$$h_\sigma^{\mathrm{III}(\Lambda\to\Lambda')}(y_\rho) := H_\sigma^{\mathrm{III}}[\xi'_\sigma, \xi_\gamma, c_{\sigma\gamma}(y_\rho)], \tag{31d}$$

where $\mu = 1, \ldots, n$, $\nu = n+1, \ldots, 2n$ and $\sigma = 2n+1, \ldots, 3n$, and definition (31$b$) is equivalent to definition (31c).

Proposition 1′ (restricted to Alice's location)**.** *Let* Λ *and* Λ^{I} *be Gaussian states of an* $n \times n \times n$ *system with vectors* $(\xi_\mu, \xi_\nu, \xi_\sigma, \eta_\lambda, \eta_\tau, \eta_\kappa)$ *and* $\left(\xi'_\mu, \xi_\nu, \xi_\sigma, \eta^{\mathrm{I}}_\lambda, \eta_\tau, \eta^{\mathrm{I}}_\kappa\right)$*, respectively, where* $\mu = 1, \ldots, n, \nu = n+1, \ldots, 2n, \sigma = 2n+1, \ldots, 3n, \lambda = 1, \ldots, (4n^2 - 2n)$, $\tau = (4n^2 - 2n + 1), \ldots, (8n^2 - 3n)$ *and* $\kappa = (8n^2 - 3n + 1), \ldots, (12n^2 - 3n)$*. If* $\Lambda \to \Lambda^{\mathrm{I}}$ *under the LOG at Alice's location, then*

(i) $\quad \xi'_\mu / \overline{\xi}_\mu \geqslant \dfrac{1}{2} \sum_{\alpha=1}^{n} \|c_{\mu\alpha}(U)\|^2_{\mathrm{F}}$,

(ii) $\quad h^{\mathrm{I}}(\Lambda \to \Lambda^{\mathrm{I}})_\mu\left(\eta^{\mathrm{I}}_\lambda\right) \geqslant 0$,

(iii) $\quad \Lambda^{\mathrm{I}}_{\mathrm{AC}} \Lambda^{-1}_{\mathrm{AC}} = \Lambda^{\mathrm{I}}_{\mathrm{AB}} \Lambda^{-1}_{\mathrm{AB}}$ (equations (10)),

where

$$\overline{\xi}_\mu = \sum_{\alpha=1}^{n} \xi_\alpha \|c_{\mu\alpha}\|^2_{\mathrm{F}} \bigg/ \sum_{\alpha=1}^{n} \|c_{\mu\alpha}\|^2_{\mathrm{F}}.$$

Proposition 2′ (restricted to Bob's location)**.** *Let* Λ^{I} *and* Λ^{II} *be Gaussian states of an* $n \times n \times n$ *system with vectors* $\left(\xi'_\mu, \xi_\nu, \xi_\sigma, \eta^{\mathrm{I}}_\lambda, \eta_\tau, \eta^{\mathrm{I}}_\kappa\right)$ *and* $\left(\xi'_\mu, \xi'_\nu, \xi_\sigma, \eta'_\lambda, \eta^{\mathrm{I}}_\tau, \eta^{\mathrm{I}}_\kappa\right)$*, respectively. If* $\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}}$ *under the LOG at Bob's location, then*

(i) $\quad \xi'_\nu / \overline{\xi}_\nu \geqslant \dfrac{1}{2} \sum_{\beta=n+1}^{2n} \|c_{\nu\beta}(U)\|^2_{\mathrm{F}} = \dfrac{1}{2} \sum_{\beta=n+1}^{2n} \|c_{\nu\beta}(V)\|^2_{\mathrm{F}}$,

(ii) $\quad h^{\mathrm{II}(\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}})}_\nu\left(\eta^{\mathrm{I}}_\lambda\right) \geqslant 0 \quad$ or $\quad h^{\mathrm{II}'(\Lambda^{\mathrm{I}} \to \Lambda^{\mathrm{II}})}_\nu\left(\eta^{\mathrm{I}}_\tau\right) \geqslant 0$,

(iii) $\quad \left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)^{-1} \Lambda^{\mathrm{II}}_{\mathrm{AB}} = \left(\Lambda^{\mathrm{I}}_{\mathrm{BC}}\right)^{-1} \Lambda^{\mathrm{II}}_{\mathrm{BC}}$ (equations (13)),

where

$$\overline{\xi}_\nu = \sum_{\beta=n+1}^{2n} \xi_\beta \|c_{\nu\beta}\|^2_{\mathrm{F}} \bigg/ \sum_{\beta=n+1}^{2n} \|c_{\nu\beta}\|^2_{\mathrm{F}}.$$

Proposition 3′ (restricted to Charlie's location)**.** *Let* Λ^{II} *and* Λ' *be Gaussian states of an* $n \times n \times n$ *system with vectors* $\left(\xi'_\mu, \xi'_\nu, \xi_\sigma, \eta'_\lambda, \eta^{\mathrm{I}}_\tau, \eta^{\mathrm{I}}_\kappa\right)$ *and* $(\xi'_\mu, \xi'_\nu, \xi'_\sigma, \eta'_\lambda, \eta'_\tau, \eta'_\kappa)$*, respectively. If* $\Lambda^{\mathrm{II}} \to \Lambda'$ *under the LOG at Charlie's location, then*

(i) $\quad \xi'_\sigma / \overline{\xi}_\sigma \geqslant \dfrac{1}{2} \sum_{\gamma=2n+1}^{3n} \|c_{\sigma\gamma}(V)\|^2_{\mathrm{F}}$,

(ii) $\quad h^{\mathrm{III}(\Lambda^{\mathrm{II}} \to \Lambda')}_\sigma\left(\eta^{\mathrm{I}}_\tau\right) \geqslant 0$,

(iii) $\quad \left(\Lambda^{\mathrm{II}}_{\mathrm{BC}}\right)^{-1} \Lambda'_{\mathrm{BC}} = \left(\Lambda^{\mathrm{II}}_{\mathrm{AC}}\right)^{-1} \Lambda'_{\mathrm{AC}}$ (equations (15)),

where

$$\overline{\xi}_\sigma = \sum_{\gamma=2n+1}^{3n} \xi_\gamma \|c_{\sigma\gamma}\|^2_{\mathrm{F}} \bigg/ \sum_{\gamma=2n+1}^{3n} \|c_{\sigma\gamma}\|^2_{\mathrm{F}}.$$

We can conclude all of the above three propositions *with the following proposition.*

Proposition 4′ ($\Lambda \to \Lambda'$ under the LOG)**.** *Let* Λ *and* Λ' *be Gaussian states of an* $n \times n \times n$ *system with vectors* (ξ_α, η_β) *and* $(\xi'_\alpha, \eta'_\beta)$*, respectively, where* $\alpha = 1, \ldots, 3n$ *and* $\beta = 1, \ldots, (12n^2 - 3n)$*. If* $\Lambda \to \Lambda'$ *under the LOG, then two points*

$$(x_\lambda) \in \left(h^{\mathrm{I}(\Lambda \to \Lambda')}_\mu\right)^{-1}(0) \cap \left(h^{\mathrm{II}(\Lambda \to \Lambda')}_\nu\right)^{-1}(0)$$

and

$$(y_\rho) \in \left(h_\nu^{\mathrm{II}'(\Lambda\to\Lambda')}\right)^{-1}(0) \cap \left(h_\sigma^{\mathrm{III}(\Lambda\to\Lambda')}\right)^{-1}(0)$$

exist such that both sets

$$S_1 = \{x_\lambda \mid g_1(x_\lambda) \leqslant \xi'_\mu/\overline{\xi}_\mu\} \cap \{x_\lambda \mid g_2(x_\lambda) \geqslant \overline{\xi}_\nu/\xi'_\nu\} \tag{32}$$

and

$$S_2 = \{y_\rho \mid g'_2(y_\rho) \leqslant \xi'_\nu/\overline{\xi}_\nu\} \cap \{y_\rho \mid g_3(y_\rho) \geqslant \overline{\xi}_\sigma/\xi'_\sigma\} \tag{33}$$

are non-empty, together with equality conditions (equations (10), (13) and (15)). Here,

$$g_1(x_\lambda) := \frac{1}{2}\sum_{\alpha=1}^{n} \|c_{\mu\alpha}(U)\|_{\mathrm{F}}^2,$$

$$g_2(x_\lambda) := 2\Bigg/\sum_{\beta=n+1}^{2n} \|c_{\nu\beta}(U)\|_{\mathrm{F}}^2,$$

$$g'_2(y_\rho) := \frac{1}{2}\sum_{\beta=n+1}^{2n} \|c_{\nu\beta}(V)\|_{\mathrm{F}}^2,$$

$$g_3(y_\rho) := 2\Bigg/\sum_{\gamma=2n+1}^{3n} \|c_{\sigma\gamma}(V)\|_{\mathrm{F}}^2.$$

Proof of proposition 1′. With the definition of matrix $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})}$ in the proof of proposition 1 (equation (23)), we have

$$H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})} = \Lambda'_{\mathrm{AA}} - \Lambda^{\mathrm{I}}_{\mathrm{AB}}\Lambda^{-1}_{\mathrm{AB}}\Lambda_{\mathrm{AA}}\left(\Lambda^{-1}_{\mathrm{AB}}\right)^T\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\right)^T + \mathrm{i}J_n - \mathrm{i}\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\Lambda^{-1}_{\mathrm{AB}}\right)J_n\left(\Lambda^{\mathrm{I}}_{\mathrm{AB}}\Lambda^{-1}_{\mathrm{AB}}\right)^T. \tag{34}$$

Note lemma 2, according to proposition 1 iff $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{II}})}\left(\Lambda^{\mathrm{II}}_{\mathrm{AB}}\right)$ is positive definite and $\Lambda^{\mathrm{I}}_{\mathrm{AB}}\Lambda^{-1}_{\mathrm{AB}} = \Lambda^{\mathrm{I}}_{\mathrm{AC}}\Lambda^{-1}_{\mathrm{AC}}$, then $\Lambda \to \Lambda^{\mathrm{I}}$ under the LOG; we then have both $H^{\mathrm{I}}_{\mu\mu} \geqslant 0$ and $\mathrm{tr}\, H^{\mathrm{I}}_{\mu\mu} \geqslant 0$, if $H^{\mathrm{I}(\Lambda\to\Lambda^{\mathrm{I}})} \geqslant 0$. With the definitions of various $\Lambda_{\alpha\beta}$ in the theorem, we can evaluate $\mathrm{tr}\, H^{\mathrm{I}}_{\mu\mu}$ and $\det H^{\mathrm{I}}_{\mu\mu}$ leading to

$$\mathrm{tr}\, H^{\mathrm{I}}_{\mu\mu} = 2\xi'_\mu - \sum_{\alpha=1}^{n} \xi_\alpha \|c_{\mu\alpha}(U)\|_{\mathrm{F}}^2 \tag{35}$$

and

$$\det H^{\mathrm{I}}_{\mu\mu} = (\xi'_\mu)^2 - \xi'_\mu \sum_{\alpha=1}^{n} \xi_\alpha \|c_{\mu\alpha}\|_{\mathrm{F}}^2 + \left|\sum_{\alpha=1}^{n} \xi_\alpha c_{\mu\alpha} c_{\alpha\mu}\right| - \left(1 - 2\sum_{\alpha=1}^{n} |c_{\mu\alpha}| + \sum_{\alpha,\gamma=1}^{n} |c_{\mu\alpha}||c_{\gamma\mu}|\right). \tag{36}$$

We have thus arrived at the statement of proposition 1′. □

One can proceed with similar steps in the proof of proposition 1′ to arrive at the conclusions of propositions 2′ and 3′. Further, by using the results of propositions 1′, 2′ and 3′, one can easily arrive at proposition 4′, the final criterion for $\Lambda \to \Lambda'$ under LOG.

5. Discussion and conclusions

Proposition 4′ constitutes the main result of the present paper. Its derivation involves a systematic reduction under the constraints required by LOG. The key intermediate step is proposition 4, which reduces the complicated conditions in the original matrix space of $\mathbf{R}^{6n\times 6n}$ to a set of much simplified problems in the matrix space of $\mathbf{R}^{2n\times 2n}$. Proposition 4′ further reduces the matrix-space criterion to a polynomial-based (parameter-space) criterion that is much more operable. It is easy to show that proposition 4′ readily recovers our previous result for this special $1\times 1\times 1$ case [7]. In comparison with bipartite cases, the additional party in tripartite Gaussian states under LOG results in additional constraints with the equalities in equations (10), (13) and (15) in proposition 4′. Obviously, these equalities have nothing to do with the complete positivity of LOG but with the intrinsic property of the multipartite Gaussian covariance matrix. The present work therefore also represents a feasible approach to arrive at the necessary criterion for the interconvertibility between all-multipartite Gaussian states.

Acknowledgments

This work is supported by the National Natural Science Foundation of China, the special funds for Major State Basic Research Project no G2001CB309500 of China and the Research Grants Council of the Hong Kong Government.

References

[1] Vogel W, Welsch D-G and Wallentowitz S 2001 *Quantum Optics, An Introduction* (Berlin: Wiley-VCH)
[2] Vaidman L 1994 *Phys. Rev.* A **49** 1473
Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
Furusawa A, Sorensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706
Bowen W P, Treps N, Buchler B C, Schnabel R, Ralph T C, Bachor H-A, Symul T and Lam P K 2003 *Phys. Rev.* A **67** 032302
[3] Duan L-M, Giedke G, Cirac J I and Zoller P 2000 *Phys. Rev. Lett.* **84** 2722
Simon R 2000 *Phys. Rev. Lett.* **84** 2726
Werner R F and Wolf M M 2001 *Phys. Rev. Lett.* **86** 3658
Giedke G, Kraus B, Lewenstein M and Cirac J I 2001 *Phys. Rev. Lett.* **87** 167904
Giedke G, Kraus B, Lewenstein M and Cirac J I 2001 *Phys. Rev.* A **64** 052303
Parker S, Bose S and Plenio M B 2000 *Phys. Rev.* A **61** 032305
Giedke G, Duan L-M, Zoller P and Cirac J I 2001 *Quantum Inf. Comput.* **1** 79
Eisert J, Simon C and Plenio M B 2002 *J. Phys. A: Math. Gen.* **35** 3911
[4] Giedke G, Wolf M M, Kruger O, Werner R F and Cirac J I 2003 *Phys. Rev. Lett.* **91** 107901
[5] Eisert J and Plenio M B 2002 *Phys. Rev. Lett.* **89** 097901
[6] Giedke G, Duan L-M, Zoller P and Cirac J I 2003 *Quantum Inf. Comput.* **3** 211
[7] Wang L, Li S-S and Zheng H-Z 2003 *Phys. Rev.* A **67** 062317
[8] Wang L, Li S-S, Yang F-H, Niu Z-C, Feng S-L and Zheng H-Z 2003 *Phys. Rev.* A **68** 020302
[9] Lindblad G 2000 *J. Phys. A: Math. Gen.* **33** 5059
[10] Horn R A and Johnson C R 1985 *Matrix Analysis* (Cambridge: Cambridge University Press)